MATH 320 NOTES, WEEK 10

Section 2.5 Change of Coordinate Matrix.

Recall that $I_V : V \to V$ is the identity linear transformation, i.e. $I_V(x) = x$ for all $x \in V$.

Theorem 1. *Suppose V is finite dimensional vector space and β, α are two bases for V. The* **change of coordinate matrix from α to β**, $Q = [I_V]_\alpha^\beta$ *is such that:*

1. *for every vector $x \in V$, $Q[x]_\alpha = [x]_\beta$,*
2. *Q is invertible and $Q^{-1} = [I_V]_\beta^\alpha$,*
3. *If $T : V \to V$ is a linear transformation, then $[T]_\alpha = Q^{-1}[T]_\beta Q$*

Proof. For simplicity, denote I_V simply by I. (1) follows since

$$Q[x]_\alpha = [I]_\alpha^\beta [x]_\alpha = [I(x)]_\beta = [x]_\beta.$$

(2): Q is invertible because I_V is invertible. Also, by 2.3,

$$[I]_\beta^\alpha Q = [I]_\beta^\alpha [I]_\alpha^\beta = [II]_\alpha = [I]_\alpha = I_n.$$

Similarly, $Q[I]_\beta^\alpha = I_n$. So $Q^{-1} = [I]_\beta^\alpha$.

(3): By 2.3, $[T]_\alpha = [ITI]_\alpha = [I]_\beta^\alpha [T]_\beta [I]_\alpha^\beta = Q^{-1}[T]_\beta Q$.

□

Computing the change of coordinate matrix:

Let $\alpha = \{x_1, ..., x_n\}$ and $\beta = \{y_1, ..., y_n\}$. Then the change of coordinate matrix Q from α to β is such that the i-th column of Q is $[x_i]_\beta$.

Example: In $\mathbb{R}^3$, let $\alpha = \{(1,1,1), (0,0,-2), (0,1,1)\}$, $\beta = \{(1,0,1), (3,0,0), (1,1,0)\}$, and $e = \{e_1, e_2, e_3\}$.

1. Let's compute the change of coordinate matrix from α to β:
   - $(1,1,1) = 1(1,0,1) - \frac{1}{3}(3,0,0) + 1(1,1,0)$
   - $(0,0,-2) = -2(1,0,1) + \frac{2}{3}(3,0,0) + 0(1,1,0)$
   - $(0,1,1) = 1(1,0,1) - \frac{2}{3}(3,0,0) + 1(1,1,0)$

   So, the matrix is $\begin{pmatrix} 1 & -2 & 1 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 1 & 0 & 1 \end{pmatrix}$

2. The change of coordinate matrix from α to e is $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 1 \end{pmatrix}$

(3) The change of coordinate matrix from e to α is $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \end{pmatrix}$

Note that $AB = BA = I_3$, i.e. $B = A^{-1}$.

(4) The change of coordinate matrix from β to e is $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

Definition 2. *Suppose that $A, B \in M_{n,n}(F)$. We say that A and B are* **similar** *if there is an invertible matrix Q such that $A = Q^{-1}BQ$. We write $A \sim B$*

As an exercise, prove the following:

Problem 1. *Being similar is an equivalence relation on matrices. I.e. it is reflexive, symmetric and transitive.*

Problem 2. *Suppose that $T : V \to V$ is a linear transformation, V is finite dimensional, and α, β are two bases for V. Then $[T]_\alpha$ and $[T]_\beta$ are similar matrices.*

Review of chapter 2

Let $T : V \to W$ be a linear transformation, V, W vector spaces over F.

(1) The dimension theorem: $\dim(V) = nullity(T) + rank(T)$.
(2) If V, W have the same finite dimension, T is one-to-one iff onto.
(3) T is invertible iff T is one–to-one and onto iff V, W are isomorphic. Then we write $V \cong W$.
(4) If $\dim(V) = n$, then $V \cong F^n$.
(5) If $\dim(V) = \dim(W)$, then $V \cong W$.

Notation and more facts:

(1) If $\dim(V) = n$ and β is a basis for V, then $[x]_\beta \in F^n$ is the coordinate vector of x relative to β, and $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ is an isomorphism (i.e. it's one-to-one, onto, and linear).
(2) If $T : V \to W$ linear, V, W finite dimensional, β is a basis for V, γ is a basis for W, then $[T]^\gamma_\beta$ is the corresponding matrix representation and for any $x \in V$,
$$[T]^\gamma_\beta [x]_\beta = [T(x)]_\gamma.$$
(3) If $T : V \to W$ and $T : W \to Z$ are linear transformation, then so is $UT : V \to Z$ defined by $UT(x) = U(T(x))$, and if α, β, γ are bases for V, U, Z, respectively, then
$$[U^\gamma_\beta][T]^\beta_\alpha = [UT]^\gamma_\alpha.$$

(4) If A is n by n and e is the standard basis for F^n, then $[L_A]_e = A$. (L_A denotes multiplication by A.)

Inverses:

If $T : V \to W$ is a one-to-one, onto linear transformation, then its **inverse** is $T^{-1} : W \to V$ defined by

$$T^{-1}(y) = x \text{ iff } T(x) = y.$$

An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix B, such that $AB = BA = I_n$. Then we write $B = A^{-1}$.

Lemma 3. *A is invertible iff L_A is invertible*

Lemma 4. *Let $\dim(V) = \dim(W) = n$, and $T : V \to W$ be a linear transformation, β any basis for V, γ any basis for W. Then T is invertible iff $[T]_\beta^\gamma$ is invertible.*

Also, in that case, if $A = [T]_\beta^\gamma$, then $A^{-1} = [T^{-1}]_\gamma^\beta$.